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Localizing the Relativistic Electron

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Abstract

A causally well-behaved solution of the localization problem for the free electron is given, with natural space-time transformation properties, in terms of Dirac's position operator \mathbf{x} . It is shown that, although

\mathbf{x} is not an observable in the usual sense, and has no positive-energy (generalized) eigenstates, the 4-vector density $(\rho(\mathbf{x}, t), \mathbf{j}(\mathbf{x}, t)/c)$ is observable, and can be localized arbitrarily precisely about any point in space, at any instant of time, using only positive-energy states. A suitable spin operator can be diagonalized at the same time.

1 Introduction

The problem of localization in the relativistic quantum mechanics of a particle with nonzero rest-mass m – whether arbitrarily precise localization is possible, and if so, how it should be described – is almost as old as relativistic quantum mechanics itself [1, 2, 3, 4, 5, 6]. Despite the efforts of many researchers over the intervening years [7, 8, 9, 10, 11, 12], the problem continues to attract much discussion [13], indicating that there is no general acceptance of any of the resolutions proposed to date.

A view sometimes expressed is that all the difficulties associated with the problem arise because any attempt to localize a particle on a scale small compared to its Compton wavelength $\lambda_C = \hbar/mc$, involves an uncertainty in energy so large that pair-production becomes possible, and a one-particle description of the physics becomes inconsistent. As was pointed out by Newton and Wigner [6] in their well-known paper on the problem, this view “really denies the possibility of the measurement of the position” of a particle. Some authors have considered it appropriate to abandon any attempt at one-particle localization, and to focus instead on local observables associated with quantized relativistic fields [12], but this approach has evidently failed to satisfy the many physicists who have continued to investigate the problem at the one-particle level [8, 9, 10, 11, 13]. Because the theory of relativity is, at its heart, a theory of relations between events in space-time, and because

it is difficult to imagine what can constitute an event other than the instantaneous localization of a particle, the denial of one-particle localizability is hard to accept.

It is important to see that the argument regarding pair-production and the Compton wavelength is inconclusive, because a single particle can have an arbitrarily large energy, and hence an arbitrarily large uncertainty in its energy. Therefore, the fact that a sufficiently small uncertainty in position implies an uncertainty in energy much greater than mc^2 does not *in itself* imply a breakdown of the one-particle picture. Indeed, Landau and Peierls [3] argued a long time ago that the minimal uncertainty in position is better represented as $\hbar c/\bar{E}$, where \bar{E} is a characteristic energy associated with the measurement, so that arbitrarily precise localization is not ruled out.

In what follows, we show that arbitrarily precise localization of a single free electron is possible, when described in terms of *observable* attributes of Dirac's position operator \mathbf{x} for the electron, in particular the familiar probability density

$$\rho(\mathbf{x}) = \psi^\dagger(\mathbf{x})\psi(\mathbf{x}) \quad (1)$$

and probability current density

$$\mathbf{j}(\mathbf{x}) = c\psi^\dagger(\mathbf{x})\boldsymbol{\alpha}\psi(\mathbf{x}) \quad (2)$$

associated with Dirac's equation. Here \mathbf{x} is the multiplicative operator acting on Dirac 4-spinor functions $\psi(\mathbf{x})$, in the Hilbert space \mathcal{H} with scalar product

$$(\psi_1, \psi_2) = \int \psi_1^\dagger(\mathbf{x})\psi_2(\mathbf{x}) d^3\mathbf{x}. \quad (3)$$

It seems the reason this simple solution has not been discovered long ago is that discussions of the localization problem have been consistently side-tracked by the following two related properties of \mathbf{x} , which appear to cause insurmountable difficulties:

(i) \mathbf{x} does not leave invariant the subspace $\mathcal{H}^{(+)} \in \mathcal{H}$ of positive-energy states, defined by

$$H\psi(\mathbf{x}) = E(\mathbf{p})\psi(\mathbf{x}), \quad (4)$$

$$H = c\boldsymbol{\alpha} \cdot \mathbf{p} + mc^2\beta. \quad (5)$$

Here $\mathbf{p} = -i\hbar\partial/\partial\mathbf{x}$, $E(\mathbf{p}) = c\sqrt{\mathbf{p}^2 + m^2c^2}$, and $\boldsymbol{\alpha}, \beta$ are the familiar 4×4 Dirac matrices. The square-root in $E(\mathbf{p})$ is defined using the Fourier transform.

(ii) \mathbf{x} has no positive-energy (generalized) eigenstates.

Because only positive-energy states are allowed to the electron, it has been common to interpret (i) to mean that \mathbf{x} cannot represent an observable, and to interpret (ii) to mean that, *á fortiori*, arbitrarily precise localization cannot be defined in terms of \mathbf{x} .

Our solution of the localization problem for the electron is based on the following two new results, which we derive below:

(I) *Despite (i), the probability density $\rho(\mathbf{x})$ and associated current density $\mathbf{j}(\mathbf{x})$ are observable quantities when ψ is a positive-energy state, as are the ‘mean \mathbf{x} -coordinates of the electron’ $\langle \mathbf{x} \rangle = (\psi, \mathbf{x}\psi)$, and the ‘uncertainty in the electron’s \mathbf{x} -coordinates,’ $\Delta_{\mathbf{x}} = \langle (\mathbf{x} - \langle \mathbf{x} \rangle)^2 \rangle^{1/2}$.*

(II) *Despite (ii), sequences of positive-energy states of the electron can be constructed for which the corresponding sequences of densities and current densities approach multiples of $\delta^3(\mathbf{x} - \mathbf{a})$ for any chosen point \mathbf{a} in space, and for which the corresponding sequences of mean values $\langle \mathbf{x} \rangle$ and uncertainties $\Delta_{\mathbf{x}}$ approach \mathbf{a} and 0, respectively.*

Positive-energy states of the electron for which the observable probability density and current density (and hence the electric charge density and current density) are arbitrarily sharply localized, and for which the observable uncertainty in the particle's \mathbf{x} -coordinates is arbitrarily small, surely describe arbitrarily precise localization of the electron. Accordingly, we conclude that the electron can indeed be localized arbitrarily sharply about any chosen point, and that localization is properly described in terms of observable properties of the Dirac operator \mathbf{x} and its associated densities. Adding force to this conclusion are the facts that such a description of localization is causally well-behaved and has natural space-time transformation properties, as we show below.

Some comments on (I) and (II) are appropriate at this point. In regard to (I), we recall that the desirability of the existence of a nonnegative probability density, with an associated current density, was one of the main motivations for Dirac's development of the equation which bears his name [1]. However, because of (i), it is by no means obvious that the Dirac densities are observable for positive-energy states of the electron. We present below a formal proof of the observability of these quantities. It is remarkable that this question does not seem to have been addressed in the past because, when multiplied by the electronic charge, (1) and (2) are the charge and electric current density of the particle, and in that form have surely been subject to much experimental scrutiny.

The nature and generality of the association of the observables of a physical system with the self-adjoint operators on a Hilbert space, has been much discussed since the earliest days of quantum mechanics [14, 1, 15]. In this connection, it is notable that $\rho(\mathbf{x})$ and $\mathbf{j}(\mathbf{x})$ are observable, for while it is true that there is a self-adjoint operator \mathbf{x} , acting in the space $\mathcal{H} \supset \mathcal{H}^{(+)}$, in terms

of which these densities are defined, this operator is not an observable in the usual sense, because of (i), and it has no (generalized) eigenstates at all in the space of physical states, because of (ii). In the case of the relativistic electron therefore, the relationship between observables and associated self-adjoint operators is less direct than in nonrelativistic quantum mechanics. In the next section, we suggest the name ‘indirect observable’ for the operator \mathbf{x} , because it is not an observable in the usual sense, but it nevertheless has observable attributes.

The result (II) is very surprising. Because \mathbf{x} has no positive-energy (generalized) eigenstates, we might reasonably expect that $\Delta_{\mathbf{x}}$ cannot be made arbitrarily small if only positive-energy states are considered. Indeed, bearing in mind the argument that is often presented regarding pair-production, we might expect that $\Delta_{\mathbf{x}} \gtrsim \lambda_C$ in positive-energy states. It seems that this has, at least implicitly, always been assumed true, but it is not so, and it is this discovery which has enabled us to construct, for the first time, the localizing sequences mentioned above.

The reader may yet feel that the non-existence of an associated self-adjoint position operator having generalized (positive-energy) eigenstates, constitutes a serious difficulty for any localization scheme defined in terms of \mathbf{x} , a difficulty which does not exist in the nonrelativistic case. However, we shall show that even in nonrelativistic quantum mechanics, localization of a particle cannot be described adequately in terms of generalized eigenstates of a position operator, but can be described in terms of localization of the probability density and current. Therefore the non-existence of generalized eigenstates in the relativistic case does not represent an insurmountable difficulty for the localization problem, as has commonly been assumed.

The most important difference between the cases of the relativistic elec-

tron and the nonrelativistic particle is not the non-existence of a position operator with generalized eigenstates in the relativistic case. Rather it is that in the nonrelativistic case, states can be constructed for which the probability density has compact support. This is not possible for the relativistic electron, as is well-known [16]; all positive-energy wavefunctions, and hence probability densities, have ‘tails’ extending to infinity in all directions of \mathbf{x} -space. These tails typically decay like $\exp(-|\mathbf{x}|/\lambda_C)$. It is an important consequence of our results that the presence of these tails does not preclude the possibility of arbitrarily precise localization using positive-energy states; in particular it does not preclude the possibility of constructing positive-energy states for which $\Delta_{\mathbf{x}}$ is arbitrarily small, and for which the probability outside any chosen compact region is arbitrarily small (*cf.* Fig. 1). We think that the presence of these tails should be accepted as an important and interesting feature of *all* positive-energy states of the electron, rather than a defect. Note that we are referring here to ‘tails’ in the space of the Dirac coordinate \mathbf{x} , not the tails in the Newton-Wigner coordinate and related concepts of localization which have been discussed, in particular by Hegerfeldt [11]. The former tails propagate causally in time, the latter do not.

There is another objection which may be raised against the use of the Dirac operator \mathbf{x} to describe localization of the electron, namely that each component \dot{x}_i , $i = 1, 2, 3$, of the associated velocity operator

$$\dot{\mathbf{x}} = c\boldsymbol{\alpha}, \quad (6)$$

has eigenvalues $\pm c$, which seems inappropriate for a massive particle. But, like \mathbf{x} , the operator \dot{x}_i does not leave $\mathcal{H}^{(+)}$ invariant. It has no positive-energy eigenstates, so that its eigenvalues cannot be observed directly. What can be observed is the expectation value of $\dot{\mathbf{x}}$, and as is well known, in any

positive-energy state ψ ,

$$\langle \dot{\mathbf{x}} \rangle = \langle c^2 \mathbf{p} / E(\mathbf{p}) \rangle. \quad (7)$$

This is appropriate for the propagation of a free, relativistic particle.

2 Electron Observables

Consider a self-adjoint operator A on the Hilbert space \mathcal{H} of 4-spinor functions. Let $P^{(+)}$ be the self-adjoint projector onto the ‘physical’ subspace of positive-energy vectors $\mathcal{H}^{(+)}$, satisfying

$$(P^{(+)})^2 = P^{(+)}, \quad (8)$$

and suppose that A is regular in the sense that

(a) for each $k = 1, 2, \dots$, the operator $P^{(+)} A^k P^{(+)}$ is self-adjoint on $\mathcal{H}^{(+)}$, and so represents an observable for the electron; and

(b) there exists a common, invariant, dense domain \mathcal{D} of vectors in $\mathcal{H}^{(+)}$ for the set of operators $P^{(+)} A^k P^{(+)}$, $k = 1, 2, \dots$

Then if the electron is in a positive-energy state $\psi \in \mathcal{D}$, the expectation value

$$\begin{aligned} \langle P^{(+)} A^k P^{(+)} \rangle &= (P^{(+)} \psi, A^k P^{(+)} \psi) \\ &= (\psi, A^k \psi) = \langle A^k \rangle \end{aligned} \quad (9)$$

is observable for every k . But knowledge of all the expectation values (moments) $\langle A^k \rangle$ is sufficient, except in pathological cases [17], to determine the distribution of probability over the spectrum of A , consistent with those moments. For example, if A has a discrete spectrum of eigenvalues a_n , $n = 1, 2, \dots$ corresponding to eigenvectors $\varphi_n \in \mathcal{H}$, then knowledge of all the

moments determines the probability $p_n = |(\phi_n, \psi)|^2$ associated with each eigenvalue a_n of A , such that $\langle A^k \rangle = \sum_n p_n (a_n)^k$, $k = 1, 2, \dots$

In this way we see that it is possible in principle, for each of a dense set of positive-energy states of the electron, to determine by measurements a corresponding distribution of probability over the spectrum of the (regular) self-adjoint operator A on \mathcal{H} , *whether or not this operator leaves $\mathcal{H}^{(+)}$ invariant*. In the discrete case, it is appropriate to call p_n the probability ‘associated with the eigenvalue’ a_n of A when the electron is in the state ψ . If A *does* leave $\mathcal{H}^{(+)}$ invariant, we can go further and call p_n ‘the probability that A will be found on measurement to have the value a_n ,’ because it is then possible that a measurement will project ψ onto ϕ_n .

Dirac’s coordinate operator \mathbf{x} is regular in the sense described. In this case, we have to show firstly that $B_{ij\dots k} = P^{(+)} x_i x_j \dots x_k P^{(+)}$ is self-adjoint, for any number of terms in the product. This can be seen by working in momentum space, where $x_i = i\hbar \partial / \partial p_i$ and $P^{(+)} = (E(\mathbf{p}) + H) / 2E(\mathbf{p})$. It is enough to see that the matrix function $P^{(+)}(\mathbf{p})$ is hermitian, and that each element of the matrix is a $C^{(\infty)}$ -function, which remains bounded as $|\mathbf{p}| \rightarrow \infty$. Then the domain of self-adjointness of $B_{ij\dots k}$ is $P^{(+)}Q \subset Q$, where $Q \subset \mathcal{H}$ is the domain of self-adjointness of $x_i x_j \dots x_k$. Secondly, we have to find a suitable common, invariant, dense domain for all operators of the form $B_{ij\dots k}$. This is provided by a space of positive-energy states, with both possible spin values, of the type described in Section 4.

It now follows that it is possible to determine by measurements the distribution of probability over the spectrum of \mathbf{x} when the electron is in (any one of a dense set of) positive-energy states, even though \mathbf{x} does not leave $\mathcal{H}^{(+)}$ invariant. In other words, the probability density (1) ‘associated with \mathbf{x} ,’ is in principle observable at any time t , and by an extension of the argument, as

a function $\rho(\mathbf{x}, t)$. Relativistic-invariance requires that this density function can be determined in any inertial frame, and since ρ is one component of the 4-vector field $(\rho, \mathbf{j}/c)$, where \mathbf{j} is defined as in (2) at each instant of time, it follows that the function $\mathbf{j}(\mathbf{x}, t)$ is also observable in principle.

Surprising observable distributions of probability over the spectrum of \mathbf{x} at any one time, that is to say some surprising observable forms for $\rho(\mathbf{x})$, lie at the heart of our solution below of the localization problem for the free electron. These distributions are arbitrarily sharply peaked about any chosen point in the spectrum of \mathbf{x} , even though \mathbf{x} has no positive-energy eigenstates.

It seems to us unnecessarily restrictive, indeed misleading, to allow the name ‘observables’ only for those A which leave $\mathcal{H}^{(+)}$ invariant and to remove the others from further consideration, given that all have observable attributes. This is particularly so in the case of \mathbf{x} , which has an important role with an intuitive meaning (as the location of charge) when the electron is coupled to an external electromagnetic field. We prefer to follow Dirac [1] and call all self-adjoint operators acting on \mathcal{H} observables. Then any observable has a (real) spectrum, and a choice of positive-energy state of the electron determines an observable distribution of probability over that spectrum. But there is a special subclass of observables, those which do leave $\mathcal{H}^{(+)}$ invariant. These have the further property that they can be diagonalized on positive-energy states, and so can be measured in the traditional sense. Accordingly, we suggest that an observable be called ‘direct’ if it leaves $\mathcal{H}^{(+)}$ invariant, and ‘indirect’ otherwise.

The reader is of course free to reject this suggestion as a matter of taste; the main point of the above argument has not been to introduce the concepts of direct and indirect observables, though we think that useful, but rather to show that quantities like the probability density (1) and current density (2),

as well as the expectation values $\langle \mathbf{x} \rangle$ and $\Delta_{\mathbf{x}}$, are observable, even though \mathbf{x} is not an observable in the traditional sense.

These ideas can obviously be extended to the case when an external field is present, so long as there is a well-defined subspace of electron states in \mathcal{H} . Whether an observable is direct or indirect may then depend on the field. For example, \mathbf{p} is direct for the free electron but indirect for the electron in a hydrogen atom, whereas \mathbf{x} is indirect in both cases. It seems to us a very interesting mathematical problem to characterise the possible *observable* distributions of probability over the spectra of important indirect observables like \mathbf{x} in such cases. For example, can the probability distribution be arbitrarily sharply peaked about any point in the spectrum of \mathbf{x} for physically-allowed states of the electron in a Coulomb field; that is to say, can the electron be localized arbitrarily sharply in such a field? Can the momentum of the electron in this case be localized arbitrarily sharply? These seem to us important questions which should now be tackled.

3 Localizing a Nonrelativistic Particle

In nonrelativistic quantum mechanics, say for a spinless particle with states described by complex functions $\chi(\mathbf{q})$, it has become common to associate localizability of the particle with the existence of a generalized eigenstate $\delta^{(3)}(\mathbf{q} - \mathbf{a})$ of the position operator \mathbf{q} , for every point \mathbf{a} in space. This is inadequate, for two reasons.

In the first place, physically realizable states must be normalized, and no sequence $\{\chi_n(\mathbf{q})\}_{n=1}^{\infty}$ of normalized states, with increasingly sharp localization of \mathbf{q} about \mathbf{a} , can approach $\delta^{(3)}(\mathbf{q} - \mathbf{a})$ as $n \rightarrow \infty$. More precisely, no such sequence can equal the generalized state $\delta^{(3)}(\mathbf{q} - \mathbf{a})$, in the sense

of the definition of generalized functions by sequences [18]. On the other hand, it is clear that a sequence of normalized states can be constructed such that the associated sequence of *densities* $\{\chi_n^*(\mathbf{q})\chi_n(\mathbf{q})\}_{n=1}^\infty$ approaches (more precisely, equals) $\delta^{(3)}(\mathbf{q} - \mathbf{a})$, with the integral over all of \mathbf{q} - space of each density in the sequence equal to 1.

In the second place, it is easily seen that it is possible to localize at any chosen time, about any chosen point, not only the particle's probability density, but also the associated current density vector

$$-\frac{i\hbar}{2m}(\chi^*(\mathbf{q})\nabla\chi(\mathbf{q}) - \chi(\mathbf{q})\nabla\chi^*(\mathbf{q})). \quad (10)$$

Consider the sequence of normalized states defined by

$$\chi_n(\mathbf{q}) = (n/\sigma\sqrt{\pi})^{3/2} e^{-n^2(\mathbf{q}-\mathbf{a})^2/(2\sigma^2)} e^{im\mathbf{v}\cdot\mathbf{q}/\hbar}, \quad (11)$$

for $n = 1, 2, \dots$, where σ , \mathbf{a} and \mathbf{v} are constants. It is easy to check that the associated sequences of densities and current densities approach (equal) $\delta^{(3)}(\mathbf{q} - \mathbf{a})$ and $\mathbf{v}\delta^{(3)}(\mathbf{q} - \mathbf{a})$, respectively. *This simultaneous localizability of probability density and current density in nonrelativistic quantum mechanics is completely obscured if localizability is associated with the generalized eigenstates of \mathbf{q} .* The densities are not defined even as generalized functions when $\chi(\mathbf{q}) = \delta^{(3)}(\mathbf{q} - \mathbf{a})$.

The importance of this fundamental point is brought home when one considers the evolution in time under Schrödinger's equation for a free particle, of a normalized state which is initially localized arbitrarily precisely, say the state (11) for some large value of n . We find that at time $t \geq 0$,

$$\chi_n^*(\mathbf{q}, t)\chi_n(\mathbf{q}, t) = \frac{n\sigma^3}{[\pi(\sigma^4 + n^4\hbar^2 t^2/m^2)]^{3/2}} e^{-n^2\sigma^2(\mathbf{q}-\mathbf{v}t)^2/[\sigma^4 + n^4\hbar^2 t^2/m^2]}. \quad (12)$$

This density is localized near $\mathbf{q} = \mathbf{a}$ at $t = 0$ and spreads out as time passes, with a centre that moves with constant velocity \mathbf{v} .

In fact, another way to describe the localizing sequence of states (11) is to say that as $n \rightarrow \infty$, $\langle \mathbf{q} \rangle \rightarrow \mathbf{a}$, $\Delta_{\mathbf{q}} \rightarrow 0$, while $\langle \mathbf{p}/m \rangle \rightarrow \mathbf{v}$. In other words, it is possible to localize the particle arbitrarily precisely while at the same time fixing its average velocity at any chosen value \mathbf{v} ; of course, the uncertainty in the velocity becomes infinitely large as $n \rightarrow \infty$. Because the velocity \mathbf{p}/m is a constant of the motion, its mean value remains constant in time, and the centre of the wavepacket moves as shown in (12).

Contrast this with the evolution in time of the generalized eigenstate $\delta^{(3)}(\mathbf{q} - \mathbf{a})$, which gives the function

$$G(\mathbf{q}, t) = \left(\frac{m}{2\pi i \hbar t} \right)^{3/2} e^{im(\mathbf{q}-\mathbf{a})^2/2\hbar t}, \quad t > 0. \quad (13)$$

At any time $t > 0$, this is centred on $\mathbf{q} = \mathbf{a}$, and is not normalizable. It is clear therefore that the generalized eigenstate $\delta^{(3)}(\mathbf{q} - \mathbf{a})$ does not even approximately describe an initial state of the particle, localized sharply near $\mathbf{q} = \mathbf{a}$, with zero or nonzero average velocity.

We conclude that, no matter how useful they may be for other purposes, generalized eigenstates of the position operator are not adequate for the description of localized states of a particle in nonrelativistic quantum mechanics.

An adequate notion of localization in the nonrelativistic case is provided by the introduction of localizing sequences of states, as follows. Bearing in mind the definition of generalized functions by sequences [18], we call a sequence $\{\chi_n\}_{n=1}^{\infty}$ of normalized states an (\mathbf{a}, \mathbf{v}) - *localizing sequence* if the associated sequences of densities and current densities approach (equal) $\delta^{(3)}(\mathbf{q} - \mathbf{a})$ and $\mathbf{v}\delta^{(3)}(\mathbf{q} - \mathbf{a})$, respectively. The sequence defined by (11) provides an example. A particle can be localized arbitrarily precisely about $\mathbf{q} = \mathbf{a}$, with its probability current localized along \mathbf{v} (or equivalently, with its mean velocity equal to \mathbf{v}), if there exists an (\mathbf{a}, \mathbf{v}) - localizing sequence of

states. If the particle has nonzero spin, localizing sequences will also carry suitable spin labels.

Note that the sharpness of localization is determined by the element of the localizing sequence which represents the state of the particle at the chosen time. Successive elements of the sequence may be thought of as arising from sharper and sharper localizations of the particle at that time. (Because the sequence of densities corresponding to a localizing sequence of states, is equal *as a whole* to a delta function [18], it is interesting to ask whether or not it may be sensible to consider a localizing sequence *as a whole* to represent an idealized, exactly localized state of the particle; but we shall not pursue this question here.)

4 Localizing the Electron

The discussion of the preceeding section shows clearly that, even in non-relativistic quantum mechanics, localization of a particle can be described adequately only in terms of arbitrarily sharply localized probability densities and currents, and not in terms of generalized eigenstates. It follows that, for the localizability of the relativistic electron at any particular instant, the important issue is not if generalized eigenstates of the Dirac operator \mathbf{x} exist – they do not – but whether or not sequences $\{\psi_n(\mathbf{x})\}_{n=1}^{\infty}$ of *positive-energy* normalized states can be found, such that the associated sequences of *observable* probability densities $\rho_n(\mathbf{x})$ and current densities $\mathbf{j}_n(\mathbf{x})$, defined as in (1) and (2), equal $\delta^{(3)}(\mathbf{x} - \mathbf{a})$ and $\mathbf{v}\delta^{(3)}(\mathbf{x} - \mathbf{a})$, respectively.

Remarkably, such (\mathbf{a}, \mathbf{v}) - localizing sequences can be found, for every point \mathbf{a} in \mathbf{x} - space, and every velocity value \mathbf{v} with $|\mathbf{v}| < c$, despite the fact that \mathbf{x} has no positive-energy generalized eigenstates. Furthermore, such

sequences can also be chosen to consist of eigenstates of a suitable spin operator. Just as in the nonrelativistic case, the sharpness of localization is determined by which element of a sequence represents the state of the electron at the chosen time. As $n \rightarrow \infty$, just as in the nonrelativistic case, $\langle \mathbf{x} \rangle \rightarrow \mathbf{a}$, $\Delta_{\mathbf{x}} \rightarrow 0$, and $\langle \dot{\mathbf{x}} \rangle \rightarrow \mathbf{v}$.

As an example, consider the sequence defined by

$$\begin{aligned}\psi_n(\mathbf{x}) &= \frac{1}{(2\pi)^{3/2}} \int \varphi_n(\mathbf{p}) e^{i\mathbf{x}\cdot\mathbf{p}} d^3p, \\ \varphi_n(\mathbf{p}) &= \frac{1}{n^{3/2}} f\left(\frac{\mathbf{p}}{n}\right) u(\mathbf{p}) e^{-i\mathbf{a}\cdot\mathbf{p}}.\end{aligned}\tag{14}$$

Here and for the remainder of this section, for simplicity of presentation we have set $\hbar = c = 1$. The function $f(\mathbf{p})$ is a complex-valued, ‘good’ function [18], that is, a C^∞ -function whose derivatives all vanish faster than any negative power of $|\mathbf{p}|$ as $|\mathbf{p}| \rightarrow \infty$. In addition, we impose

$$\int |f(\mathbf{p})|^2 d^3p = 1, \quad \int |f(\mathbf{p})|^2 \frac{\mathbf{p}}{|\mathbf{p}|} d^3p = \mathbf{v}.\tag{15}$$

The spinor $u(\mathbf{p})$ in (14) satisfies

$$Hu(\mathbf{p}) = E(\mathbf{p})u(\mathbf{p}), \quad u^\dagger(\mathbf{p})u(\mathbf{p}) = 1,\tag{16}$$

and consequently the first of the conditions (15) ensures that φ_n and hence ψ_n is normalized for every n . We can further require that u , and hence each ψ_n , is an eigenspinor of some suitable spin operator with eigenvalue $+(1/2)$ or $-(1/2)$, which commutes with \mathbf{p} and H , say for definiteness the third component of Pryce’s spin operator [4]

$$\begin{aligned}\tilde{S}_3(\mathbf{p}) &= U(\mathbf{p})\left(-\frac{i}{2}\alpha_1\alpha_2\right)U^\dagger(\mathbf{p}), \\ U(\mathbf{p}) &= (E(\mathbf{p})I_4 + H\beta)/\mathcal{E}(\mathbf{p}),\end{aligned}\tag{17}$$

where $\mathcal{E}(\mathbf{p}) = \sqrt{2E(\mathbf{p})(E(\mathbf{p}) + m)}$, and I_4 is the 4×4 unit matrix

As the Fourier transform of $\psi_{\mathbf{n}}(\mathbf{x})$ is $\varphi_{\mathbf{n}}(\mathbf{p})$, so the transform of $\psi_{\mathbf{n}}^\dagger(\mathbf{x})$ is $\varphi_{\mathbf{n}}^\dagger(-\mathbf{p})$, and the transform of $\rho_{\mathbf{n}}(\mathbf{x})$ (resp. $j_{ni}(\mathbf{x})$, $i = 1, 2$ or 3) is the convolution

$$\begin{aligned} & \frac{1}{(2\pi)^{3/2}} \int \varphi_{\mathbf{n}}^\dagger(\mathbf{q} - \mathbf{p}) Q \varphi_{\mathbf{n}}(\mathbf{q}) d^3 q \\ & \quad (= \frac{e^{-i\mathbf{a} \cdot \mathbf{p}}}{(2\pi)^{3/2}} R_{\mathbf{n}}(\mathbf{p}), \text{ say}), \end{aligned} \quad (18)$$

where $Q = I_4$ (resp. α_i). Noting that the transform of $\delta^{(3)}(\mathbf{x} - \mathbf{a})$ is $e^{-i\mathbf{a} \cdot \mathbf{p}}/(2\pi)^{3/2}$, we have to show that $\{R_{\mathbf{n}}(\mathbf{p})\}_{\mathbf{n}=1}^\infty$ equals 1 (resp. v_i) as a generalized function. From (14) and (18) we have

$$\begin{aligned} R_{\mathbf{n}}(\mathbf{p}) &= \frac{1}{n^3} \int [f(\frac{\mathbf{q} - \mathbf{p}}{n}) u(\mathbf{q} - \mathbf{p})]^\dagger Q f(\frac{\mathbf{q}}{n}) u(\mathbf{q}) d^3 q \\ &= \int f^*(\mathbf{r} - \frac{\mathbf{p}}{n}) f(\mathbf{r}) u^\dagger(n\mathbf{r} - \mathbf{p}) Q u(n\mathbf{r}) d^3 r. \end{aligned} \quad (19)$$

By Taylor's Theorem,

$$\begin{aligned} f^*(\mathbf{r} - \frac{\mathbf{p}}{n}) &= f^*(\mathbf{r}) - \frac{p_j}{n} f_j^*(\mathbf{r} - \eta \frac{\mathbf{p}}{n}), \\ u_a^\dagger(n\mathbf{r} - \mathbf{p}) &= u_a^\dagger(n\mathbf{r}) - p_k u_{ka}^\dagger(n\mathbf{r} - \theta \mathbf{p}), \end{aligned} \quad (20)$$

where u_a^\dagger , $a = 1, 2, 3, 4$, are the components of u^\dagger , $f_j(\mathbf{s}) = \partial f(\mathbf{s})/\partial s_j$, $u_{ka}^\dagger(\mathbf{s}) = \partial u_a^\dagger(\mathbf{s})/\partial s_k$, and η, θ are some functions of \mathbf{r}, \mathbf{p} and n (and a , in the case of θ) satisfying $0 \leq \eta \leq 1$, $0 \leq \theta \leq 1$. In (20), the summation convention applies to the repeated subscripts j and k . Substituting (20) in (19), we get

$$\begin{aligned} R_{\mathbf{n}}(\mathbf{p}) &= \int |f(\mathbf{r})|^2 u^\dagger(n\mathbf{r}) Q u(n\mathbf{r}) d^3 r \\ &\quad - \frac{p_j}{n} \int f_j^*(\mathbf{r} - \frac{\mathbf{p}}{n}) f(\mathbf{r}) u^\dagger(n\mathbf{r}) Q u(n\mathbf{r}) d^3 r \\ &\quad - p_k \int |f(\mathbf{r})|^2 u_k^\dagger(n\mathbf{r} - \theta \mathbf{p}) Q u(n\mathbf{r}) d^3 r \\ &\quad + \frac{p_j p_k}{n} \int f_j^*(\mathbf{r} - \frac{\mathbf{p}}{n}) f(\mathbf{r}) u_k^\dagger(n\mathbf{r} - \theta \mathbf{p}) Q u(n\mathbf{r}) d^3 r \end{aligned}$$

$$\begin{aligned}
&= A_n - \frac{p_j}{n} B_{nj}(\mathbf{p}) \\
&\quad - p_k C_{nk}(\mathbf{p}) + \frac{p_j p_k}{n} D_{njk}(\mathbf{p}), \quad \text{say.}
\end{aligned} \tag{21}$$

Making a standard choice of Dirac matrices [1] with β diagonal, we get for the eigenspinor of $\tilde{S}_3(\mathbf{s})$ with eigenvalue $+1/2$,

$$u^\dagger(\mathbf{s}) = (E(\mathbf{s}) + m, 0, s_3, s_1 - i s_2) / \mathcal{E}(\mathbf{s}), \tag{22}$$

and it is easily checked that

$$\begin{aligned}
|u_a^\dagger(\mathbf{s})| &\leq 1, \quad |u_{ka}^\dagger(\mathbf{s})| < \frac{2}{m}, \\
|u_{ka}^\dagger(\mathbf{s})| &< \frac{2}{|\mathbf{s}|} \quad (\mathbf{s} \neq \mathbf{0}),
\end{aligned} \tag{23}$$

and indeed that the magnitudes of all derivatives of u_a^\dagger , of all orders, are bounded by constants. It follows successively [18] that $\varphi_n(\mathbf{p})$, $\psi_n(\mathbf{x})$, $\rho_n(\mathbf{x})$, $\mathbf{j}_n(\mathbf{x})$ and $R_n(\mathbf{p})$ are good functions for every n .

Using the first and second of (23), we see that $B_{nj}(\mathbf{p})$ and $D_{njk}(\mathbf{p})$ are bounded by constants independent of \mathbf{p} and n . In $C_{nk}(\mathbf{p})$, we note that if $n|\mathbf{r}| > 2|\mathbf{p}|$ ($> 2\theta|\mathbf{p}|$), then $(n|\mathbf{r}| - \theta|\mathbf{p}|) > \frac{1}{2}n|\mathbf{r}|$, and so using the third of (23),

$$|u_{ka}^\dagger(n\mathbf{r} - \theta\mathbf{p})| < \frac{2}{|n\mathbf{r} - \theta\mathbf{p}|} < \frac{4}{n|\mathbf{r}|} \tag{24}$$

whenever $n|\mathbf{r}| > 2|\mathbf{p}|$. Accordingly, we write

$$\begin{aligned}
C_{nk}(\mathbf{p}) &= \left(\int_{n|\mathbf{r}| < 2|\mathbf{p}|} + \int_{n|\mathbf{r}| > 2|\mathbf{p}|} \right) \\
&\quad \times |f(\mathbf{r})|^2 u_k^\dagger(n\mathbf{r} - \theta\mathbf{p}) Q u(n\mathbf{r}) d^3 r \\
&= C_{nk}^<(\mathbf{p}) + C_{nk}^>(\mathbf{p}), \quad \text{say.}
\end{aligned} \tag{25}$$

Then, using (23) and (24),

$$\begin{aligned}
|C_{nk}^>(\mathbf{p})| &< \frac{\text{const.}}{n} \int_{n|\mathbf{r}|>2|\mathbf{p}|} \frac{|f(\mathbf{r})|^2}{|\mathbf{r}|} d^3r \\
&< \frac{\text{const.}}{n} \int \frac{|f(\mathbf{r})|^2}{|\mathbf{r}|} d^3r < \frac{\text{const.}}{n}.
\end{aligned} \tag{26}$$

Furthermore, we have from (23) that

$$\begin{aligned}
|C_{nk}^<(\mathbf{p})| &< \text{const.} \int_{|\mathbf{r}|<\frac{2|\mathbf{p}|}{n}} |f(\mathbf{r})|^2 d^3r \\
&< \text{const.} \frac{|\mathbf{p}|^3}{n^3}.
\end{aligned} \tag{27}$$

Finally, in (21), we see that when $Q = 1$ we get $A_n = 1$ because of the first of (15); and when $Q = \alpha_i$, we have $u^\dagger(n\mathbf{r})\alpha_i u(n\mathbf{r}) = nr_i/E(n\mathbf{r})$, so that

$$A_n = \int |f(\mathbf{r})|^2 \frac{r_i}{\sqrt{|\mathbf{r}|^2 + (m^2/n^2)}} d^3r, \tag{28}$$

which approaches v_i because of the second of (15). From these results for A_n , $B_{nj}(\mathbf{p})$, $C_{nk}(\mathbf{p})$ and $D_{njk}(\mathbf{p})$, it is easy to check that $\{R_n(\mathbf{p})\}_{n=1}^\infty$ is a regular sequence [18] which, as a generalized function, equals 1 (resp. v_i) as required.

The choice $f(\mathbf{p}) = (1/mc\sqrt{\pi})^{3/2} \exp(-\mathbf{p}^2/2m^2c^2)$ in (14) leads to a $(0,0)$ -localizing sequence of positive-energy states, and a corresponding sequence of spherically-symmetric probability densities $\rho_n(r)$, $r = |\mathbf{x}|/\lambda_C$. Fig. 1 shows these increasingly localized densities for $n = 5, 7, 10$, as determined numerically with the help of *Mathematica* [19]. Localization well within the Compton wavelength ($r = 1$) is evident, even for such small values of n .

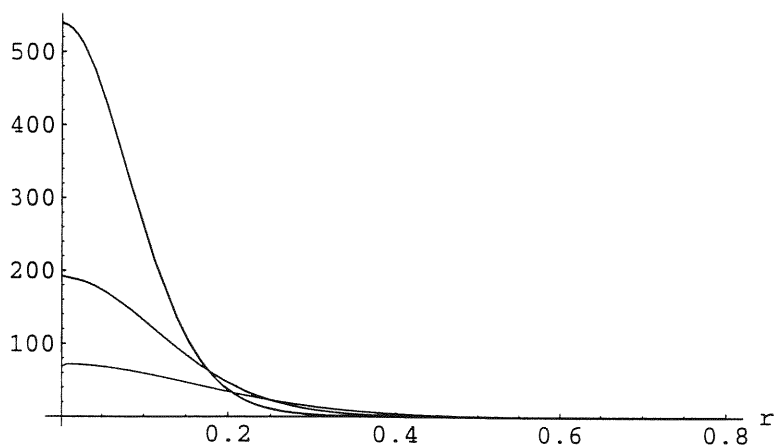


Fig. 1

5 Space-time Transformations, Causality and Orthogonality

Localizing sequences transform naturally under the action of the (extended) Poincaré group. This is a consequence of the covariant transformation properties of (ct, \mathbf{x}) and $(\rho, \mathbf{j}/c)$.

Thus it is easily seen from (14) that a spatial translation by \mathbf{b} of (each element of) an (\mathbf{a}, \mathbf{v}) - localizing sequence produces an $(\mathbf{a} + \mathbf{b}, \mathbf{v})$ - localizing sequence; that the parity operation produces a $(-\mathbf{a}, -\mathbf{v})$ - localizing sequence; that the operation of time-reversal produces an $(\mathbf{a}, -\mathbf{v})$ - localizing sequence; and that the result of a rotation by \mathbf{R} produces an $(\mathbf{R}\mathbf{a}, \mathbf{R}\mathbf{v})$ - localizing

sequence.

Lorentz boosts and translations in time require a little more discussion. Consider for example the effect of a boost along the 3-axis, with $x'_3 = x_3 \cosh \sigma + ct \sinh \sigma$ and $ct' = ct \cosh \sigma + x_3 \sinh \sigma$. In classical, relativistic physics, if a point-particle with charge e is, in the original frame, at the point \mathbf{a} , with velocity \mathbf{v} , at (on) the instant $t = 0$, then in the transformed frame, the particle is at the point $(x'_1, x'_2, x'_3) = (a_1, a_2, a_3 \cosh \sigma)$, with velocity $\mathbf{v}' = (c/(c \cosh \sigma + v_3 \sinh \sigma))(v_1, v_2, v_3 \cosh \sigma + c \sinh \sigma)$, on the hyperplane $ct' = x'_3 \tanh \sigma$. It follows that the corresponding charge and current densities are given at $t = 0$ in the first frame by

$$\mathbf{j}(0, \mathbf{x}) = \mathbf{v}\rho(0, \mathbf{x}) = e\mathbf{v}\delta^{(3)}(\mathbf{x} - \mathbf{a}), \quad (29)$$

and on the hyperplane $ct' = x'_3 \tanh \sigma$ in the transformed frame by

$$\begin{aligned} \mathbf{j}'((x'_3/c) \tanh \sigma, x'_1, x'_2, x'_3) &= \mathbf{v}'\rho'((x'_3/c) \tanh \sigma, x'_1, x'_2, x'_3) = \\ &= (v_1, v_2, c \sinh \sigma + v_3 \cosh \sigma) \cosh \sigma \delta(x_1 - a_1) \delta(x_2 - a_2) \delta(x_3 - a_3 \cosh \sigma). \end{aligned} \quad (30)$$

This is mirrored in the effect of the boost on an (\mathbf{a}, \mathbf{v}) - localizing sequence at (on) the instant $t = 0$. From the transformation laws for a 4-vector field, we see that as the associated sequence of $(\rho, \mathbf{j}/c)$ values at $t = 0$ approaches (equals) $(1, \mathbf{v}/c)\delta^{(3)}(\mathbf{x} - \mathbf{a})$, in an obvious notation, then the sequence of transformed values $(\rho', j'_1/c, j'_2/c, j'_3/c)$, evaluated on the hyperplane $ct' = x'_3 \tanh \sigma$, approaches (equals)

$$\begin{aligned} &(c \cosh \sigma + v_3 \sinh \sigma, v_1, v_2, c \sinh \sigma + v_3 \cosh \sigma) \\ &\times \frac{\cosh \sigma}{c} \delta(x_1 - a_1) \delta(x_2 - a_2) \delta(x_3 - a_3 \cosh \sigma). \end{aligned} \quad (31)$$

In regard to the evolution of a localizing sequence in time, we note that, as a consequence of (1) and (2), and the fact that the eigenvalues of each

α_i are ± 1 , the Dirac densities satisfy the inequality $|\mathbf{j}(\mathbf{x}, t) \cdot \mathbf{n}| \leq c\rho(\mathbf{x}, t)$, where \mathbf{n} is any constant unit vector. Since the velocity of probability flow is $\mathbf{j}(\mathbf{x}, t)/\rho(\mathbf{x}, t)$, this inequality is a necessary and sufficient condition for the spread of the probability density ρ in every direction to occur at speeds no greater than the speed of light. Thus the localization scheme is also guaranteed to behave causally.

The detailed analysis of how localizing sequences of positive-energy states evolve in time under Dirac's equation is an interesting and nontrivial problem to which we hope to return in future work, together with an analysis of how the spin variable labelling a localizing sequence transforms under rotations and Lorentz boosts.

We note finally that localizing sequences of positive-energy states constructed as above satisfy a limiting notion of orthogonality: if $\{\psi_n\}_{n=1}^\infty$ is an (\mathbf{a}, \mathbf{v}) - localizing sequence and $\{\psi'_n\}_{n=1}^\infty$ is an $(\mathbf{a}', \mathbf{v}')$ - localizing sequence, then it is easily seen from the second of (14), with the help of the Riemann-Lebesgue Lemma, that

$$\lim_{n \rightarrow \infty} (\psi_n, \psi'_n) = 0 \quad \text{when} \quad \mathbf{a} \neq \mathbf{a}'. \quad (32)$$

States from two sequences labelled by different spin eigenvalues are of course exactly orthogonal to each other.

6 Concluding Remarks

We have shown that the Dirac electron can be localized arbitrarily sharply about any point in space, at any chosen instant, when localization is described in terms of *observable* attributes of Dirac's operator \mathbf{x} .

This description of arbitrarily precise (but not exact) localization may be compared with the notion of *exactly* localized states introduced by New-

ton and Wigner [6]. These arise when one first identifies an appropriate space of physical states for a free, relativistic particle (more precisely, for an elementary system), and then seeks to find generalized states satisfying certain conditions which are appropriate when the particle is exactly localized. In the case of the electron, one obtains as a result the generalized eigenstates of the Pryce-Newton-Wigner position operator, which leaves the space of physical states invariant. Because this space carries an irreducible, unitary representation R (say) of the extended Poincaré group, the operator obtained, and the whole Newton-Wigner concept of localization, have an invariant group-theoretic meaning. This was emphasized by Bacry [9], who showed that the Pryce-Newton-Wigner position operator can be expressed in terms of the Poincaré group generators. In this sense, the Newton-Wigner description of localization is independent of the *realization* of the representation R , and there is no particular significance attaching to the realization in terms of Dirac spinors.

In contrast, out of all manifestly covariant realizations of R with multi-component wavefunctions $\psi(\mathbf{x}, t)$, only Dirac's realization has an associated nonnegative probability density $\rho(\mathbf{x}, t)$ and associated current density $\mathbf{j}(\mathbf{x}, t)$ satisfying a conservation equation [10]. These densities play a crucial role in our localization scheme, so that Dirac's realization is distinguished from all others when the localization problem is considered in the manner which we have advocated in this paper.

In our view, localization of the free electron can only be described in terms of Dirac's equation and its associated dynamical variables. Any other realization of R is unitarily equivalent mathematically to that carried by the positive-energy subspace $\mathcal{H}^{(+)} \subset \mathcal{H}$, but is *not equivalent to it physically*: the point is that the unitary mapping from the one to the other is nonlocal in

every case [20]. It is partly for this reason that investigations of the localization problem within the abstract framework of an irreducible representation of the Poincaré group have not led to the solution that we have described.

There is a deeper reason: many investigations of this type have assumed from the outset that the central problem is to find a self-adjoint operator acting in the space of physical states, which can be used to define localization on compact regions, or exact localization at a point in terms of generalized eigenstates. But in the case of the relativistic electron, unlike the case of a nonrelativistic particle, the physical space of states is a different subspace of \mathcal{H} for each different external field. The free particle is just one special case of this. It is for this reason that *all* selfadjoint operators on \mathcal{H} must be regarded as having physical significance; as we have shown, any one of them may have observable attributes even in cases when it does not leave the subspace of physical states invariant, and so is not an observable in the usual sense. Our solution of the localization problem depends on this more subtle relationship between operators and observables which exists for the Dirac electron.

The noncausal properties of Newton-Wigner localization are well-known [7, 8, 11]. A particle localized in the Newton-Wigner sense at the origin at time $t = 0$, can be found at $t > 0$, according to the same notion of localization, outside the sphere of radius ct centred at the origin. This is unacceptable in a relativistic theory. It is also known that states exactly localized in the Newton-Wigner sense do not transform in a simple way with respect to Lorentz boosts – in short, because the Pryce-Newton-Wigner operator is not the three-vector part of a four-vector.

It is pleasing therefore that we have been able to show that there does exist, at least for the electron, a causally well-behaved localization scheme, with natural space-time transformation properties. We have emphasized

above that our analysis has depended critically on the particular structure of Dirac's equation, and we see no reason to expect that it will extend to other particles described by other wave equations, excepting those for the 2- and 4-component neutrinos.

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References

- [*] On leave from: Department of Mathematics, University of Queensland, Brisbane 4072, Australia.
- [1] P.A.M. Dirac, *The Principles of Quantum Mechanics*, 4th Ed. (Clarendon Press, Oxford, 1958).
- [2] E. Schrödinger, Sitzungsber. Preuss. Akad. Wiss. Phys-Math. Kl. **XXIV**, 418 (1930); **III**, 63 (1931).
- [3] L. Landau and R. Peierls, Z. Phys. **69**, 56 (1931), transl. in: D. ter Haar (Ed.), *Collected papers of L.D. Landau* (Pergamon, Oxford, 1965).
- [4] M.H.L. Pryce, Proc. Roy. Soc. London **150A**, 166 (1935); **195A**, 62 (1948).
- [5] A. Papapetrou, Praktika Acad. Athènes **15**, 404 (1940); C. Møller, Commun. Dublin Inst. Adv. Stud. **A**, No. 5 (1949).
- [6] T.D. Newton and E.P. Wigner, Rev. Mod. Phys. **21**, 400 (1949).
- [7] A.S. Wightman and S. Schweber, Phys. Rev. **98**, 812 (1955); A.S. Wightman, Rev. Mod. Phys. **34**, 845 (1962).

- [8] L.L. Foldy and S.A. Wouthuysen, Phys. Rev. **78**, 29 (1950); S. Tani, Prog. Theoret. Phys. **6**, 267 (1951); D.G. Currie, T.F. Jordan and E.C.G. Sudarshan, Rev. Mod. Phys. **35**, 350 (1963); T.F. Jordan and N. Mukunda, Phys. Rev. **132**, 1842 (1963); W. Weidlich and A.K. Mitra, Nuovo Cim. (10) **30**, 385 (1963); T.O. Philips, Phys. Rev. **136** B893 (1964); G.N. Fleming, Phys. Rev. **137**, B188 (1965); J. Math. Phys. **7**, 1959 (1966); R.A. Berg, J. Math. Phys. **6**, 34 (1965); A. Galindo, Nuovo Cim. (10) **37**, 413 (1965); A.J. Kálnay, "The Localization Problem," in M. Bunge (Ed.) *Studies in the Foundations, Methodology and Philosophy of Science*. Vol. 4, *Problems in the Foundations of Physics* (Springer Verlag, Berlin, 1971); A.J. Kálnay and P.L. Torres, Phys. Rev. **D7**, 1707 (1973); G.C. Hegerfeldt Phys. Rev. **D10** 3320 (1974); Z. Haba, Nuovo Cim. A(11)**32**, 174 (1976); B.K. Skagerstam, Int. J. Theoret. Phys. **15**, 213 (1976); J.F. Perez and I.F. Wilde, Phys. Rev. **D16**, 315 (1977); E.P. Wigner and R.F. O'Connell, Phys. Lett. **61A**, 353 (1977); Phys. Lett. **67A** 319 (1978).
- [9] H. Bacry, J. Math. Phys. **5**, 109-111 (1964).
- [10] A.O. Barut and S. Malin, Rev. Mod. Phys. **40**, 632-650 (1968).
- [11] G.C. Hegerfeldt and S.N.M. Ruijsenaars, Phys. Rev. **D22**, 377 (1980); G.C. Hegerfeldt, Phys. Rev. Lett. **54**, 2395 (1985); Nucl. Phys. B Proc. Suppl. **6**, 231 (1989); Ann. Phys. (Leipzig) **7**, 716 (1998).
- [12] J.M. Knight, J. Math. Phys. **2**, 459 (1964); A.L. Licht, J. Math. Phys. **4**, 1443 (1963); R. Haag, *Local Quantum Physics* (Springer-Verlag, New York, 1992).

- [13] H. Bacry, *Localizability and Space in Quantum Physics* (Springer-Verlag, Berlin, 1988); V.V. Dodonov and S.S. Mizrahi, Phys. Letts **177A**, 394 (1993); M-T. Jaekel and S. Reynaud, Phys. Lett. **A220**, 10 (1996); R. Omnes, J. Math. Phys. **38**, 708 (1997); S.T. Ali, Int. J. Theoret. Phys. **37**, 365 (1998).
- [14] E. Schrödinger, *The Interpretation of Quantum Mechanics: Dublin Seminars (1949-1955) and Other Unpublished Essays*, Ed. and with Intro. by M. Bitbol (Ox Bow Press, Woodbridge, Conn., 1995).
- [15] E.P. Wigner, Z. Phys. **133**, 101 (1952).
- [16] B. Thaller, *The Dirac Equation* (Springer-Verlag, New York, 1992).
- [17] J.A. Shohat and J.D. Tamarkin, *The Problem of Moments* (Amer. Math. Soc., New York, 1943).
- [18] M.J. Lighthill, *Introduction to Fourier Analysis and Generalized Functions* (Cambridge University Press, 1958).
- [19] *Mathematica*, Wolfram Research Inc., Champaign, Illinois, USA.
- [20] U. H. Niederer and L. O’Raifeartaigh, Fortschr. der Physik **22**, 111 (1974).